Electromagnetic field equations for anisotropic media

1971 J. Phys. A: Gen. Phys. 4 197

(http://iopscience.iop.org/0022-3689/4/2/003)

J. Phys. A: Gen. Phys., 1971, Vol. 4. Printed in Great Britain

Electromagnetic field equations for anisotropic media

S. J. LEWANDOWSKI†
Instytut Fizyki PAN, Warszawa, Zielna 37, Poland
MS. recieved 29th September 1970

Abstract. The tensor identity

$$\boldsymbol{\alpha} \,.\, \operatorname{curl} \bar{\boldsymbol{\alpha}} \,.\, \operatorname{curl} \boldsymbol{F} = |\boldsymbol{\alpha}|(\operatorname{grad} \operatorname{div} \boldsymbol{\alpha}^{-1} \,.\, \boldsymbol{F} - \operatorname{div} \boldsymbol{\alpha}^{-1} \,.\, \operatorname{grad} \boldsymbol{F})$$

where $\boldsymbol{\alpha}$ is a constant and nonsingular matrix and $\boldsymbol{F}$ is a properly regular vector field, is derived and used to discuss the analogies between second order equations for fields in an anisotropic medium and the Helmholtz equation. The scalar equation for fields in uniform media is also considered; it is shown that this equation is of fourth order and for plane waves in an unbounded medium it is reduced to the generalized Fresnel equation.

1. Introduction

Electromagnetic fields in anisotropic media are described by complicated sets of second order differential equations. As yet there exists no general theory of these equations, but many individual problems were thoroughly investigated, in particular those concerning wave propagation in magnetized matter (cf. Lax and Button 1962).

The experience gained so far shows that, apart from basic theoretical difficulties, the manipulation of lengthy initial equations is cumbersome and technically difficult. This is so even if only one of the constitutive parameters of the medium is a tensor, not to mention more general cases, which are also interesting and not devoid of some practical significance—for example, solid state maser cavities contain doubly anisotropic media whose electric and magnetic anisotropy axes usually do not coincide (Siegman 1964).

In this paper we present a modified form of the field equations for a general doubly anisotropic medium. This form allows one to avoid some of the technical difficulties mentioned above. Moreover, it is a direct continuation of the Helmholtz equation used in the isotropic case and therefore it gives better understanding of the similarities and differences between field description in isotropic and anisotropic media.

2. Second order equations

Let us consider a uniform medium characterized by constitutive tensors $\boldsymbol{\epsilon}$ and $\boldsymbol{\mu}$. Assuming that no free charges and currents are present, the amplitudes $\boldsymbol{E}$ and $\boldsymbol{H}$ of fields dependent on time as $\exp(-\mathrm{i}\omega t)$ are solutions of the following second order equations (Epstein 1956):

$$\operatorname{curl} \boldsymbol{\mu}^{-1} \,.\, \operatorname{curl} \boldsymbol{E} = \omega^2 \boldsymbol{\epsilon} \,.\, \boldsymbol{E} \qquad \operatorname{curl} \boldsymbol{\epsilon}^{-1} \,.\, \operatorname{curl} \boldsymbol{H} = \omega^2 \boldsymbol{\mu} \,.\, \boldsymbol{H}. \tag{1}$$

Observe that the divergence equations

$$\operatorname{div} \boldsymbol{\epsilon} \,.\, \boldsymbol{E} = 0 \qquad \operatorname{div} \boldsymbol{\mu} \,.\, \boldsymbol{H} = 0 \tag{2}$$

are identically fulfilled by $\boldsymbol{E}$ and $\boldsymbol{H}$ and in the present case there is no need to consider them separately.

† Present address: Laboratoire de Résonance Magnetique, Faculté des Sciences de Paris, Quai St Bernard, Paris V, France.

For an isotropic medium ($\boldsymbol{\epsilon} = \epsilon \mathbf{I}$, $\boldsymbol{\mu} = \mu \mathbf{I}$; $\mathbf{I}$ denotes the identity matrix) equations (1) are usually replaced by the Helmholtz equations

$$\nabla^2 \boldsymbol{F} + \omega^2 \epsilon \mu \boldsymbol{F} = 0 \tag{3}$$

where $\boldsymbol{F}$ stands for $\boldsymbol{E}$ or $\boldsymbol{H}$. The transformation follows from the identity

$$\operatorname{curl}\operatorname{curl} \boldsymbol{F} = \operatorname{grad}\operatorname{div} \boldsymbol{F} - \nabla^2 \boldsymbol{F} \tag{4}$$

and from equations (2). Therefore (3) is not equivalent to (1); to obtain the equivalence, equations (2) must be imposed on (3). It is interesting, however, that in some special problems the solutions of (3) with nonvanishing divergence cannot be neglected (cf. Slater 1950).

We intend to show that in the anisotropic case considered here an analogue of equation (3) can be constructed.

Instead of identity (4) we now use its generalization (see Appendix):

$$\bar{\boldsymbol{\alpha}} \,.\, \operatorname{curl} \boldsymbol{\alpha} \,.\, \operatorname{curl} \boldsymbol{F} = |\boldsymbol{\alpha}|(\operatorname{grad}\operatorname{div} \bar{\boldsymbol{\alpha}}^{-1} \,.\, \boldsymbol{F} - \operatorname{div} \bar{\boldsymbol{\alpha}}^{-1} \,.\, \operatorname{grad} \boldsymbol{F}) \tag{5}$$

which is valid for any nonsingular matrix of constant coefficients $\boldsymbol{\alpha}$ and any twice differentiable (with continuous derivatives) vector field $\boldsymbol{F}$; $\bar{\boldsymbol{\alpha}}$ denotes here the transpose of $\boldsymbol{\alpha}$ and $|\boldsymbol{\alpha}| = \det \boldsymbol{\alpha}$. To emphasize the similarities between (4) and (5) we introduce the shorthand notation

$$\operatorname{div} \boldsymbol{\alpha} \,.\, \operatorname{grad} \boldsymbol{F} = \nabla \,.\, \boldsymbol{\alpha} \,.\, \nabla \boldsymbol{F} = \nabla_\alpha^2. \tag{6}$$

The affinity between ∇^2 and ∇_α^2 is rather obvious; note that in the rectangular coordinate system $\{x_i\}$ we have

$$\nabla_\alpha^2 \boldsymbol{F} = \sum_{i,k} \alpha_{ik} \frac{\partial^2 \boldsymbol{F}}{\partial x_i \partial x_k} = \nabla_{\bar{\alpha}}^2 \boldsymbol{F} \tag{7}$$

as compared with

$$\operatorname{grad}\operatorname{div} \boldsymbol{\alpha} \,.\, \boldsymbol{F} = \nabla\nabla \,.\, \boldsymbol{\alpha} \,.\, \boldsymbol{F} = \operatorname{grad} \sum_{k,k'} \alpha_{kk'} \frac{\partial F_{k'}}{\partial x_k}. \tag{8}$$

Let us note also that if $\nabla\nabla \,.\, \boldsymbol{\alpha}$ is treated as a symbolic matrix, then

$$\nabla_\alpha^2 = \operatorname{Tr}(\nabla\nabla \,.\, \boldsymbol{\alpha}). \tag{9}$$

In § 3 we prove another important property of this matrix: it is singular:

$$\det(\nabla\nabla \,.\, \boldsymbol{\alpha}) = 0. \tag{10}$$

From (1) and (5) we immediately obtain our result. The equation for the electric field $\boldsymbol{E}$, for instance, now takes the form (Lewandowski 1965)

$$\nabla_\mu^2 \boldsymbol{E} - \operatorname{grad}\operatorname{div} \bar{\boldsymbol{\mu}} \,.\, \boldsymbol{E} + \omega^2 \boldsymbol{\kappa} \,.\, \boldsymbol{E} = 0 \tag{11}$$

where

$$\boldsymbol{\kappa} = |\mu| \bar{\boldsymbol{\mu}}^{-1} \,.\, \boldsymbol{\epsilon}.$$

It will be seen later that $\boldsymbol{\kappa}$ is an important quantity. It appears that whether the medium can be considered as isotropic or uniaxial depends not so much on $\boldsymbol{\epsilon}$ and $\boldsymbol{\mu}$ alone as on their 'ratio' given by $\boldsymbol{\kappa}$.

Although there is a very striking resemblance between equations (3) and (11), an important difference should also be pointed out. We were not able to incorporate in (11) the information given by the divergence relations (2). Therefore, equations (1) and (11) are exactly equivalent.

It follows that the relation

$$\operatorname{div} \boldsymbol{\epsilon} \,.\, \boldsymbol{E} = \operatorname{div} \bar{\boldsymbol{\mu}} \,.\, \boldsymbol{\kappa} \,.\, \boldsymbol{E} = 0$$

must be satisfied identically by the solutions of (11). Indeed, let us apply $\operatorname{div} \bar{\boldsymbol{\mu}}$. to (11) and use definition (6):

$$\operatorname{div} \bar{\boldsymbol{\mu}} \,.\, \nabla_{\mu}{}^{2} \boldsymbol{E} - \nabla_{\mu}{}^{2} \operatorname{div} \bar{\boldsymbol{\mu}} \,.\, \boldsymbol{E} + \omega^{2} \operatorname{div} \bar{\boldsymbol{\mu}} \,.\, \boldsymbol{\kappa} \,.\, \boldsymbol{E} = 0.$$

It remains to observe that $\nabla_{\mu}{}^{2}$ and $\operatorname{div} \bar{\boldsymbol{\mu}}$. commute.

Equation (11) can be used as a convenient algorithm for differential operations indicated in (1), since all vector products of the involved quantities are here replaced by cartesian (scalar or dyadic) products.

Another advantage is that (11) allows us to draw immediately certain conclusions which are not so apparent from (1). For instance, let us observe that the fields are governed by the Helmholtz equation not only in the case of an isotropic medium but also in the quasi-isotropic case of a medium whose constitutive tensors are both proportional to the same Hermitian matrix $\boldsymbol{\nu}$. Then

$$\boldsymbol{\kappa} = |\boldsymbol{\nu}|\epsilon\mu \mathbf{I}$$

and (11) is reduced to

$$\nabla_{\nu}{}^{2} \boldsymbol{E} + \omega^{2} |\boldsymbol{\nu}| \epsilon\mu \boldsymbol{E} = 0.$$

$\nabla_{\nu}{}^{2}$ can be transformed to ∇^{2} by proper choice of the coordinate system $\{x_i\}$ (see Waldron 1957).

3. The fourth order equation

Combining equations (7), (8) and (11) we can write the following system of second order partial differential equations with constant coefficients

$$\sum_{k=1}^{3} L_{ik} E_k = 0 \qquad i = 1, 2, 3 \tag{12}$$

where

$$L_{ik} = \nabla_{\mu}{}^{2} \delta_{ik} - \frac{\partial}{\partial x_i} \sum_{k'} \mu_{kk'} \frac{\partial}{\partial x_{k'}} + \omega^{2} \varkappa_{ik} \tag{13}$$

and δ_{ik} is the Kronecker symbol.

In each of equations (12) appear all three unknown scalar functions E_k. Epstein (1956) and also Bochenek (1961) raised an interesting question: Is it possible to obtain separate equations for each individual field component E_k? They investigated only the special case of a gyrotropic (i.e. uniaxial) medium and upon successive elimination of unknown functions they found that all E_k (and H_k) are solutions of the same fourth order equation. We intend to show that the above statement is true for *any* uniform anisotropic medium.

From (12) it is clear (cf. Courant and Hilbert 1962) that the arbitrary field component E_k must indeed satisfy a separate differential equation of an order not higher than sixth, given by

$$LE_k = 0 \tag{14}$$

where L is the symbolic determinant of the matrix (L_{ik})

$$L = \det(L_{ik}). \tag{15}$$

Straightforward, if rather tedious, calculations show that the operation L is of fourth order.

We first prove that L cannot be of sixth order. Observe that the highest order term in L is given by the symbolic determinant

$$D = \det(\nabla_{\mu}{}^{2}\mathbf{I} - \nabla\nabla\,.\,\bar{\boldsymbol{\mu}}) \tag{16}$$

and we wish to prove $D = 0$. Let us denote for brevity the matrix elements of $\nabla\nabla\,.\,\bar{\boldsymbol{\mu}}$ by d_{ik}:

$$d_{ik} = (\nabla\nabla\,.\,\bar{\boldsymbol{\mu}})_{ik} = \frac{\partial}{\partial x_i}\sum_{k'} \mu_{kk'}\frac{\partial}{\partial x_{k'}}.$$

Then we can write the following useful property of these elements

$$d_{ik}d_{mn} = d_{in}d_{mk}$$

from which it follows that all second degree minors $M_{pq}(\nabla\nabla\,.\,\bar{\boldsymbol{\mu}})$ of $\nabla\nabla\,.\,\bar{\boldsymbol{\mu}}$ vanish

$$\pm M_{pq}(\nabla\nabla\,.\,\bar{\boldsymbol{\mu}}) = d_{ik}d_{mn} - d_{in}d_{mk} = 0 \tag{17}$$

and hence also

$$\det(\nabla\nabla\,.\,\bar{\boldsymbol{\mu}}) = 0. \tag{18}$$

Now, for any 3×3 matrix $\boldsymbol{\alpha}$ and number λ we have (Mostowski and Stark 1963)

$$\det(\boldsymbol{\alpha} - \lambda\mathbf{I}) = \det\boldsymbol{\alpha} - \lambda\sum_{p} M_{pp}(\boldsymbol{\alpha}) + \lambda^2 \mathrm{Tr}\,\boldsymbol{\alpha} - \lambda^3. \tag{19}$$

Using this relation for $\boldsymbol{\alpha} = \nabla\nabla\,.\,\boldsymbol{\mu}$ and $\lambda = \nabla_{\mu}{}^{2}$ and taking into account (9), (17) and (18) we obtain $D = 0$, *QED*.

To show that L must be of fourth order we require the explicit form of this operation. Further calculations, based on relation (19) and recorded elsewhere (Lewandowski 1969), yield

$$L = \nabla_{\epsilon}{}^{2}\nabla_{\mu}{}^{2} + \omega^2\nabla_{\eta}{}^{2} + \omega^4|\boldsymbol{\epsilon}||\boldsymbol{\mu}| \tag{20}$$

where

$$\nabla_{\eta}{}^{2} = |\boldsymbol{\epsilon}|[\nabla_{\mu}{}^{2}\mathrm{Tr}(\boldsymbol{\epsilon}^{-1}\,.\,\bar{\boldsymbol{\mu}}) - \nabla^{2}_{\bar{\boldsymbol{\mu}}\cdot\boldsymbol{\epsilon}^{-1}\cdot\bar{\boldsymbol{\mu}}}]$$

or

$$\boldsymbol{\eta} = |\boldsymbol{\epsilon}|(\bar{\boldsymbol{\mu}}\mathrm{Tr}\,\boldsymbol{\epsilon}^{-1}\,.\,\bar{\boldsymbol{\mu}} - \bar{\boldsymbol{\mu}}\,.\,\boldsymbol{\epsilon}^{-1}\,.\,\bar{\boldsymbol{\mu}}). \tag{21}$$

The fourth order terms in (20), given by the product $\nabla_{\boldsymbol{\epsilon}}{}^{2}\nabla_{\mu}{}^{2}$, cannot vanish since $\nabla_{\boldsymbol{\alpha}}{}^{2} = 0$ implies $\boldsymbol{\alpha}$ identically equal to zero (see definition (7)). Thus equation (14) must be always of fourth order.

It is interesting that the first and the last term in (20) is invariant under the transformation

$$\boldsymbol{\mu} \Rightarrow \boldsymbol{\epsilon}. \tag{22}$$

If the middle term has the same property it means that equation (14) is satisfied also by the cartesian components H_k of the magnetic field since the Maxwell equations are invariant under the transformation $\boldsymbol{E} \Rightarrow \boldsymbol{H}$, $\boldsymbol{\mu} \Rightarrow -\boldsymbol{\epsilon}$. To check this possibility we rewrite (21) as (Lewandowski 1969)

$$\boldsymbol{\eta} = |\bar{\boldsymbol{\mu}}+\boldsymbol{\epsilon}|(\boldsymbol{\epsilon}^{-1}+\bar{\boldsymbol{\mu}}^{-1})^{-1}-|\boldsymbol{\mu}|\boldsymbol{\epsilon}-|\boldsymbol{\epsilon}|\bar{\boldsymbol{\mu}}. \tag{23}$$

Clearly, $\boldsymbol{\eta}$ is transposed if subjected to (22), but this does not affect ∇_η^2. Therefore, we can give to (14) a more general meaning

$$LX = 0 \qquad X = E, H. \tag{24}$$

It must be understood that the solutions of this equation constitute a wider class than the solutions of initial equations (1) (Courant and Hilbert 1962). Equivalence of both problems can be achieved only by retaining with (24) at least one of the lower-order scalar equations, for example, one of equations (12).

4. 'Diagonal' anisotropy

One of the obvious lines of further research is to see if L, as given by (20) and (21) or (23), can be factorized into two second order operations. In general this is not a very simple task and we limit ourselves only to a hypothetical medium with 'diagonal' anisotropy, that is, a medium for which there exists such a rectangular coordinate system $\{x_i\}$ that in it both $\boldsymbol{\mu}$ and $\boldsymbol{\epsilon}$ are diagonal.

Assume therefore

$$\epsilon_{ik} = \epsilon_i\delta_{ik} \qquad \mu_{ik} = \mu_i\delta_{ik}$$

Then

$$\nabla_\eta^2 = \sum_{i=1}^{3} (\mu_{i+1}\epsilon_{i+2}+\mu_{i+2}\epsilon_{i+1})\mu_i\epsilon_i\frac{\partial^2}{\partial x_i^2} \tag{25}$$

where we adopt the periodicity convention of index i:

$$i+3 \equiv i.$$

Clearly, if for some $i = s$ there is

$$\mu_{s+1}\epsilon_{s+2} = \mu_{s+2}\epsilon_{s+1} = a \tag{26}$$

then

$$\nabla_\eta^2 = a(\mu_s\nabla_\epsilon^2+\epsilon_s\nabla_\mu^2)$$

and

$$L = (\nabla_\mu^2+\omega^2a\mu_s)(\nabla_\epsilon^2+\omega^2a\epsilon_s). \tag{27}$$

In this case the solutions of equation (24) in an unbounded region can be constructed as linear combinations of solutions of two independent Helmholtz equations. If $\boldsymbol{\mu} = \mu\mathbf{I}$, we easily recognize that one of these equations, corresponding to the first factor in (27), describes what in optics is called the 'ordinary wave', and the other one—the 'extraordinary wave' (cf. Landau and Lifshitz 1960).

Probably the assumption of diagonal anisotropy and condition (26) represent the most general situation in which it is possible to factorize L. A similar result was obtained earlier by Bochenek (1961) under a stronger assumption of uniaxial symmetry:

$$\epsilon_{s+1} = \epsilon_{s+2} \qquad \mu_{s+1} = \mu_{s+2}.$$

This indicates that (26) should be recognized as a generalized condition of uniaxial anisotropy. In fact the matrix $\boldsymbol{\kappa}$ appearing in (11) is, under this condition, uniaxial, as can be easily verified.†

† A. Kujawski (1971 private communication) recently proved that to factorize L it is sufficient to assume that $\boldsymbol{\kappa}$ is uniaxial.

5. Fresnel's equation. Application to rectangular cavities

Let us examine now if plane waves

$$U \sim e^{i\boldsymbol{k}.\boldsymbol{r}}$$

can be the solutions of equation (24). For such functions we have identically

$$\nabla_\alpha^2 = -K_\alpha^2(\boldsymbol{k}) \tag{28}$$

where

$$K_\alpha^2(\boldsymbol{k}) = \sum_{i,i'} \alpha_{ii'} k_i k_{i'}$$

is a bilinear form of the cartesian components k_i of the wave vector $\boldsymbol{k}$. Therefore (24) is reduced to the ordinary algebraic equation (see also Fedorov 1958)

$$K_\epsilon^2 K_\mu^2 - \omega^2 K_\eta^2 + \omega^4 |\boldsymbol{\epsilon}||\boldsymbol{\mu}| = 0 \tag{29}$$

which is the necessary and sufficient condition that U describes a plane wave propagating in the medium with (given) angular frequency ω. Substituting $\boldsymbol{\mu} = \mu \mathbf{I}$ we obtain from (29) the familiar Fresnel equation.

Equation (29) can also be utilized in a slightly less orthodox manner. Let there be an ordered set of numbers

$$(k_1, k_2, k_3)$$

assumed to be the rectangular components of wave vector $\boldsymbol{k}$, and let us find the angular frequency ω of the plane wave associated with these numbers.† Obviously the problem has two solutions, ω_1 and ω_2, given by the general formula

$$\omega_{1,2}^2 = \frac{1}{2|\boldsymbol{\epsilon}||\boldsymbol{\mu}|}[K_\eta^2 \mp \{(K_\eta^2)^2 - 4|\boldsymbol{\epsilon}||\boldsymbol{\mu}|K_\epsilon^2 K_\mu^2\}^{1/2}]. \tag{30}$$

In the case of diagonal anisotropy this can be written more compactly (Lewandowski 1965)

$$\omega_{1,2}^2 = \frac{1}{2|\boldsymbol{\epsilon}||\boldsymbol{\mu}|}\left\{\sum_i \gamma_{+i}^2 \mp \left(\sum_{i,i'} (i,i')\gamma_{-i}^2 \gamma_{-i}^2\right)^{1/2}\right\} \tag{30a}$$

where

$$\gamma_{\pm i}^2 = (\mu_{i+1}\epsilon_{i+2} \pm \mu_{i+2}\epsilon_{i+1})\mu_i \epsilon_i k_i^2$$

and the bracket (i, i') is defined by

$$(i,i') = \begin{cases} +1 & \text{for } i = i' \\ -1 & \text{for } i \neq i'. \end{cases}$$

If for some value s of the index i we have $\gamma_{-s}^2 = 0$, that is, either if condition (26) is satisfied or $k_s = 0$, we obtain from (30*a*), putting if required $k_s^2 = 0$,

$$\begin{aligned} \omega_1^2 &= \frac{k_s^2}{a} + \frac{k_{s+1}^2}{\epsilon_s \mu_{s+2}} + \frac{k_{s+2}^2}{\mu_{s+1}\epsilon_s} = \frac{K_\epsilon^2}{a s_s} \\ \omega_2^2 &= \frac{k_s^2}{a} + \frac{k_{s+1}^2}{\mu_s \epsilon_{s+2}} + \frac{k_{s+2}^2}{\epsilon_{s+1}\mu_s} = \frac{K_\mu^2}{a\mu_s}. \end{aligned} \tag{30b}$$

The last equalities above follow directly from (27).

† The author is indebted for this idea and for the first draft of equation (30*a*) to Mr J. Zagrodzinski.

In the isotropic case, (30) is further reduced to

$$\omega_1^2 = \omega_2^2 = \frac{1}{\epsilon\mu} \sum_i k_i^2. \tag{30c}$$

An identical formula is used for the determination of resonant frequencies of a rectangular cavity with perfectly conducting walls, only the k_i are then given a different meaning:

$$k_i = \frac{m_i \pi}{l_i} = k_{m_i} \tag{31}$$

where l_i is the cavity length and m_i is a natural number.

The observed analogy is not accidental. It is caused by the fact that among the general solutions of equation (24), for which the operation ∇_α^2 is equivalent to multiplication by $-K_\alpha^2$, there are not only travelling waves $e^{i\boldsymbol{k}.\boldsymbol{r}}$ but also standing waves of the form

$$\prod_{i=1}^{3} (A_i e^{ik_i x_i} \pm B_i e^{-ik_i x_i})$$

and these can be made to fit the appropriate homogeneous boundary conditions at surfaces x_i = constant. In particular, this holds for the functions

$$U_{mi} = \cos(k_{m_i} x_i) \sin(k_{m_{i+1}} x_{i+1}) \sin(k_{m_{i+2}} x_{i+2}) \tag{32}$$

to which are proportional the rectangular components E_{mi} of the electric field inside an isotropically filled cavity:

$$E_{mi} = A_{mi} U_{mi}. \tag{33}$$

The subscript m denotes here the ordered set of three numbers (m_1, m_2, m_3) appearing in (31).

Let us suppose for a while that the electric field in a rectangular cavity containing an anisotropic medium is also given by (32) and (33), differing from the isotropic solution only in the amplitudes A_{mi}. The resonant frequencies would then be given by (30). Hence, for each value of index m, that is, for each set of functions U_{mi}, there should be at least two sets of amplitudes, $\{A_{mi}\}_1$ and $\{A_{mi}\}_2$, generating two solutions, $\boldsymbol{E}_{m1}$ and $\boldsymbol{E}_{m2}$, of Maxwell's equations. The $\boldsymbol{E}_{m1}$ and $\boldsymbol{E}_{m2}$ fields would thus correspond to (normally degenerate) TE and TM modes in an isotropically filled cavity.

Unfortunately, the above assumption is true only for 'diagonal' filling of the cavity (anisotropy axes coinciding with the coordinate axes set by the boundary conditions). However, the conclusion that the introduction of an anisotropic medium lifts the degeneracy of resonant modes is generally true and formula (30), for the case of an arbitrary cavity filling, yields the first order perturbational approximation of the resonant frequencies (Lewandowski 1969).

6. Conclusions

Our main concern in this paper was with what may be called the external shape of the field equations. Possible applications and consequences of our formulation were not pursued. It should be pointed out, however, that the introduction of our operations ∇_μ^2 = div $\bar{\mu}$. grad and grad div $\bar{\mu}$ could be helpful in the study of boundary value problems for closed regions.

As to the fourth order equation (24), it appears to be far too difficult for any attempts at direct solution. Therefore, its usefulness is probably limited to the approximate solution of certain eigenvalue problems, as demonstrated in § 5.

Acknowledgments

The author is grateful to Mr J. Zagrodzinski for many stimulating discussions and helpful comments.

Appendix

In the proof of identity (5) we use index notation and the summation convention. Together with the usual vector product of vectors $\boldsymbol{A}$ and $\boldsymbol{B}$

$$(\boldsymbol{A}\times\boldsymbol{B})_r = A_s B_t \epsilon_{str}$$

where ϵ_{str} is the Ricci permutation symbol, we shall require vector products of a dyadic $\boldsymbol{\alpha}$ and vector $\boldsymbol{B}$, defined by

$$(\boldsymbol{\alpha}\times\boldsymbol{B})_{qt} = \alpha_{qr} B_s \epsilon_{rst} \tag{A1}$$

$$(\boldsymbol{B}\times\boldsymbol{\alpha})_{qt} = \epsilon_{qrs} B_r \alpha_{st}. \tag{A2}$$

We introduce also the abbreviation $\partial/\partial x_i = d_i$, so that the usual differential operations will be written now as

$$\begin{aligned} \operatorname{div}\boldsymbol{F} &= \nabla\,.\,\boldsymbol{F} = d_s F_s \qquad (\operatorname{grad}\boldsymbol{F})_{rs} = (\nabla\boldsymbol{F})_{rs} = d_r F_s \\ (\operatorname{div}\boldsymbol{\alpha})_k &= (\nabla\,.\,\boldsymbol{\alpha})_k = d_i \alpha_{ik} \qquad (\operatorname{curl}\boldsymbol{F})_r = (\nabla\times\boldsymbol{F})_r = \epsilon_{rst} d_s F_t \end{aligned} \tag{A3}$$

and the operations involved in (5) will take the form

$$\nabla_{\alpha}{}^2 = \operatorname{div}\boldsymbol{\alpha}\,.\,\operatorname{grad}\boldsymbol{F} = d_i \alpha_{ik} d_k F_r \tag{A4}$$

$$(\operatorname{grad}\operatorname{div}\boldsymbol{\alpha}\,.\,\boldsymbol{F})_r = d_r d_s \alpha_{st} F_t = d_s \alpha_{st} d_r F_t = (\operatorname{div}\boldsymbol{\alpha}\,.\,\overline{\operatorname{grad}\boldsymbol{F}})_r. \tag{A5}$$

(i) Consider now the expression ($\boldsymbol{\alpha}$ constant)

$$[\boldsymbol{\alpha}\,.\,(\nabla\times\boldsymbol{F})]_q = \alpha_{qr}\epsilon_{rst} d_s F_t = d_s \epsilon_{str} \bar{\alpha}_{rq} F_t = d_s(\boldsymbol{F}\times\bar{\boldsymbol{\alpha}})_{sq}.$$

This can be written as

$$\boldsymbol{\alpha}\,.\,(\nabla\times\boldsymbol{F}) = \operatorname{div}(\boldsymbol{F}\times\bar{\boldsymbol{\alpha}}). \tag{A6}$$

(ii) Let us compute

$$[(\boldsymbol{\alpha}\,.\,\boldsymbol{B})\times\boldsymbol{\alpha}]_{qt} = \epsilon_{qrs} \alpha_{rp} B_p \alpha_{st}. \tag{A7}$$

But (Brand 1947)

$$|\boldsymbol{\alpha}|\epsilon_{npt} = \epsilon_{qrs}\alpha_{qn}\alpha_{rp}\alpha_{st}$$

and if we multiply this equality by $\alpha_{nm}{}^{-1}$ and sum over m, we obtain

$$|\boldsymbol{\alpha}|\epsilon_{npt}\alpha_{nm}{}^{-1} = \epsilon_{qrs}\alpha_{rp}\alpha_{st}\delta_{qm}.$$

With this result (A7) becomes

$$[(\boldsymbol{\alpha}\,.\,\boldsymbol{B})\times\boldsymbol{\alpha}]_{qt} = |\boldsymbol{\alpha}|\epsilon_{npt}\alpha_{nq}{}^{-1}B_p$$

or, from (A1)

$$[\boldsymbol{\alpha}\,.\,\boldsymbol{B}]\times\boldsymbol{\alpha} = |\boldsymbol{\alpha}|\bar{\boldsymbol{\alpha}}^{-1}\times\boldsymbol{B}. \tag{A8}$$

(iii) Finally, we consider

$$[\boldsymbol{\beta}\times(\nabla\times\boldsymbol{F})]_{qt} = \beta_{qr}\epsilon_{snp} d_n F_p \epsilon_{rst} = \beta_{qr}(d_t F_r - d_r F_t)$$

or, with the aid of (A5),

$$\boldsymbol{\beta} \times (\nabla \times \boldsymbol{F}) = \boldsymbol{\beta} \cdot (\overline{\nabla \boldsymbol{F}} - \nabla \boldsymbol{F}). \tag{A9}$$

Applying successively (A6), (A8) and (A9) to the expression

$$\boldsymbol{\alpha} \cdot [\nabla \times \{\bar{\boldsymbol{\alpha}} \cdot (\nabla \times \boldsymbol{F})\}]$$

we obtain immediately identity (5).

References

BOCHENEK, K., 1961, *Analiza Pol Elektromagnetycznych w Ośrodkach Żyromagnetycznych* (Warszawa: PAN).

BRAND, L., 1947, *Vector and Tensor Analysis* (New York: Wiley), p. 329.

COURANT, R., and HILBERT, D., 1962, *Methods of Mathematical Physics*, Vol. II (New York: Interscience), p. 14.

EPSTEIN, P. S., 1956, *Rev. mod. Phys.*, **28**, 3.

FEDOROV, F. I., 1958, *Optics of Anisotropic Media* (Minsk: MIR), in Russian.

LANDAU, L. D., and LIFSHITZ, E. M., 1960, *Electrodynamics of Continuous Media* (Oxford: Pergamon), pp. 313–43.

LAX, B., and BUTTON, K. J., 1962, *Microwave Ferrites and Ferrimagnetics* (New York: McGraw-Hill).

LEWANDOWSKI, S. J., 1965, *Proc. 4th Natn. Conf. on Electromagnetic Theory* (Warszawa: PAN), in Polish.

—— 1969, *DSc Thesis*, Instytut Fizyki PAN, Warsaw.

MOSTOWSKI, A., and STARK, M., 1963, *Elementy Algebry Wyższej* (Warszawa: PWN), p. 107.

SIEGMAN, A. E., 1964, *Microwave Solid State Masers* (New York: McGraw-Hill).

SLATER, J. C., 1950, *Microwave Electronics* (London: Van Nostrand), pp. 57–83.

WALDRON, R. A., 1957, *Proc. IEE*, **104B**, suppl. 6, 307.